Volume of Solids – *Khātavyavahāra* in Sanskrit Texts

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Abstract

Mathematics in ancient India evolved and developed for practical purposes. For Vedic seers performing *Yajñas* (sacrifices) was important and for this purpose, fire altars were to be constructed. Knowledge of the construction of three-dimensional figures helped the seers to raise them. As it was necessary to calculate the cubical content of the ditch to be dug for the basement of the buildings, temples, fire altars etc., the excavations and cubical content of solids (*Khāta-vyavahāra*) is treated as an individual topic in the mathematical texts of ancient India. The present topic is discussed considering these texts in order. The rules and methods of finding the volume of several three-dimensional solids such as pyramid, frustum, cone, sphere etc. were dealt with in several ancient texts. An attempt is made to bring out these results to the knowledge of the general public.

Key words: Excavations, volume, solids, *khāta*, *upapatti*.

Abbreviations:

Āryabhaṭīya – Ā; *Brāhma-Sphuṭa-Siddhānta* – Br.Sp.Si; *Buddhivilāsinī* – BV; *Gaṇita-Sāra-Saṅgraha* – GSS; *Līlāvatī* – L

Introduction

The texts like *Śulbasūtras*, *Āryabhaṭīya* (*Ā*) of Āryabhaṭa I, *Brāhma-Sphuṭa-Siddhānta* (*Br.Sp.Si*) of Brahmagupta, *Gaṇita-Sāra-Saṅgraha (GSS)* of Mahāvīra, *Līlāvatī (L)* of Bhāskara II and many other texts deal with the calculation of volume of the solids and the excavations, which are considered here for discussions. *Śulbasūtras* give measures and constructions of several three-dimensional fire-altars, bricks etc. for which an example is given here. In *Āryabhaṭīya* (*Ā*), the approximate volume of a right pyramid and a sphere is given. Brahmagupta gave the correct formulae for regular and non-uniform solids and also for a frustum. Vīrasena has mentioned a rule for the volume of a trapezoidal solid. Śrīdhara has given an expression for the volume of the frustum of a cone and an approximate formula for the volume of a sphere. Mahāvīra gives a rule for arriving at the cubical contents of excavations and also for one having in the middle a tapering projection, volumes of frustum-like solids, sphere (approximate) and for a triangular pyramid. Bhāskara II gives the volume of an irregular ditch, volume of a pyramid and its frustum, volumes of a ditch with square or circular base, a cone and a sphere. Gaṇeśa Daivajña and Jyeṣṭhadeva later provided *upapattis* (demonstrations/proofs) for the rules of Bhāskara.

1. Śulbasūtras

Śulbasūtras (earlier than 800 BC) are the earliest Indian texts available at present that contain mathematical content. *Śulbasūtrakāras* were familiar with the concept of volume since they used to fix the height as well as the number of layers and total number of bricks in the fire altars of regular geometrical shape. For example, the construction of *śmaśānaciti* (pyre) is explained in *Baudhāyana-Śulbasūtras* (III. 253–269).

श्मशानचितं चिन्वितेति विज्ञायते । ...ताभिश्चतस्रो वा नव वा चतुर्दश वा चितिरुपाधाय शेषमवाञ्चमक्ष्णयापच्छिन्द्यात् । अर्धमुद्धारेत् ॥ तस्य नित्यो विभागो यथायोगमिष्टकानां ह्रासवृद्धिः ॥

"According to tradition, a fire-altar in the form of a *śmaśānaciti* is to be constructed … With the bricks, 4 or 9 or 14 layers are made, the remaining layer is diagonally cut in the downward directions and half of it removed. Its division is exact. Larger and smaller bricks are taken according as these fit (1; pp.35-6, 97-8)".

The interpretation of this is as follows: The above *sutras* enjoin that the *śmaśānaciti* fireplace should have a base of an isosceles trapezium and its top surface should slope from one edge (eastern) to the other (western) so that its eastern height is up to the neck and the western height is up to the navel; yet the volume of the fire-place is to be the same as that of the usual fire-place. Datta [2; p.103] gave a modern representation that this construction is based on the approximate formula for the frustum of a pyramid namely,

$V = \{(a + a')/2\}\{(b + b')/2\}h$, where (a,b) and (a',b') are the dimensions of the lower and upper faces.

2. Āryabhaṭa I

"Half of that area (of the triangular base) multiplied by the height is the volume of a six-edged solid [3; p.39]."

Area of six-edged solid = ½ × area of base triangle × height of the solid.

(b) Āryabhaṭa gives formulae for the area of a circle and approximate volume of a sphere (*Ā*. II.7.):

समपरिणाहस्यार्धं विष्कम्भार्धहतमेव वृत्तफलम्।
तन्निजमूलेन हतं घनगोलफलं निरवशेषम्॥

"Half of the circumference, multiplied by the semi-diameter certainly gives the area of a circle. That area (of the section through diameter) multiplied by its own square root gives the volume of the sphere [3; p.40]".

The formulae that are given here:

Area of a circle $= (C/2)\,(d/2)$

Volume of sphere $= (C/2)\,(d/2)\sqrt{\frac{C}{2} \times \frac{d}{2}}$

$\Rightarrow \quad = \pi r^2 \sqrt{\pi r^2}$ (where $d = 2r$ i.e. diameter = twice the radius)

$= \sqrt{\pi}\,\pi r^3$

Note: The formulae for the six-edged solid and the volume of the sphere given in *Ā* are approximate and these are upgraded and perfected by the later mathematicians.

3. Brahmagupta

Brahmagupta, the most celebrated mathematician belonging to the School of Ujjain, wrote his *Brāhma-Sphuṭa-Siddhānta* (*Br.Sp.Si.*) in 629 CE. This is also a book on astronomy, but chapters XII and XVIII are on mathematics.

(a) He has given the accurate formula for the volume of a pyramid (*Br.Sp.Si*.XII.44):

क्षेत्रफलं वेधगुणं समखातफलं हृतं त्रिभिः सूच्याः।
मुखतलतुल्यभुजैक्यान्येकाग्रहृतानि समरज्जुः॥

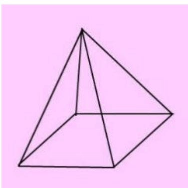


Fig. 1: Pyramid

"The volume of a pit of uniform depth is area of *samakhāta* multiplied by the depth. This divided by three is the volume of a *sūcī*, a figure tapering to a point. In an excavation having the same breadth at the face and bottom, the aggregates (of the partial products of lengths and depths) divided by the total (length) will be the mean measure (*samarajju*) of the depth [6; pp.135-36]."

The following formulae are given in this verse:

(i) The volume of a pit of uniform depth = area of section × depth
(ii) The volume of a solid which tapers uniformly = (1/3) area of base to a point *i.e.* that of a cone (*sūcī*) ×depth
(iii) The volume of a pyramid = (1/3) × the volume of a prism on the same base
(iv) When a pit has non-uniform depth, this is divided into parallel strips of uniform depth. The product of the breadths when divided by the sum of the widths of these strips is the *samarajju* (average depth).

(b) The volume of a frustum is given for the first time by Brahma Gupta (*Br.Sp.Si*.XII.45, 46) [7]:

मुखतलयुतिदलगुणितं वेधगुणं व्यावहारिकं गणितम्।
मुखतलगणितैक्यार्धं वेधगुणं स्यात् गणितमौत्रम्॥
औत्रगणिताद्विशोध्य व्यवहारफलं भवेत्त्रिभिः शेषम्।
लब्धं व्यवहारफले प्रक्षिप्य भवति फलं सूक्ष्मम्॥

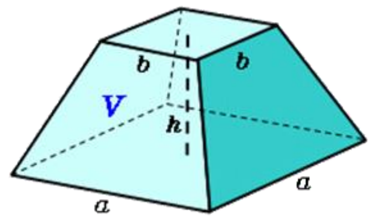


Fig. 2: Frustum

"The square of half the sum of the sides in the face and the base multiplied by the depth is the *Vyavahārika* volume. Half the sum of the areas of the face and base multiplied by the depth is the *Autra* volume. The *Vyavahārika* volume (V_v) is subtracted from the *Autra* volume (V_a) and being divided by three; this quotient is added to the *Vyavahārika* volume. This gives the exact volume (V) [6]".

This can be expressed in modern notation as follows:

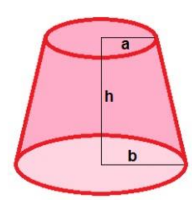
$$V = V_v + (1/3)\,(V_a - V_v)$$

$$= \{(a + b)/2\}^2 \times h + [\{(a^2+b^2)/2\} - \{(a+ b)/2\}^2]\ \ (h/3)$$

$$= (h/3)\,(a^2 + b^2 + ab)$$

4. Vīrasena

Vīrasena (710 – 790 CE) has written a commentary on the *Ṣaṭkhaṇḍāgama*, [8] called *Dhavalā*. In this work he has quoted an old *Karaṇagātha*, for the volume of a trapezoidal solid:



मुखतलसमासार्द्धं उत्सेधगुणं च वेधेन।
घनगणितं जानीयात् वेत्रासनसंस्थिते क्षेत्रे॥

"Half the sum of the face and the base multiplied by the height and by the depth is to be known as the volume of a figure resembling a rattan seat".

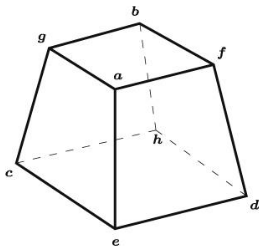


Fig. 3: Trapezoid

The volume of a trapezoidal solid = Area of the trapezium × thickness
= (1/2) (face + base) × height × depth

Vīrasena details a method of infinite division for finding the volume of the frustum of a cone in his *Dhavalā.* If a and b are the diameters at the base and at the top and h, the height of the frustum, the volume of the frustum is
$= (\pi \cdot h/4)(1/3)(a^2 - ab + b^2)$

5. Śrīdhara

(a) Śrīdhara (8th century CE) has written several works in astronomy and mathematics. Among them *Siddhāntaśekhara*, *Pāṭīgaṇita* and *Trirāśikā* are well-known. Śrīdhara has given an expression for the volume of the frustum of a cone (*Triśatikā*, Rule 38).

Fig. 4: Frustum

मुखतलतद्योगानां वर्गैक्यकृतेः पदं दशगुणायाः।
वेधगुणं चतुरन्वितविंशतिभक्तं फलं कूपे॥

"The square of the diameter of the base, the square of diameter of face and square of sum of these two diameters, these three squares are added and this sum is squared,this is multiplied by ten. Take square-root of this result. Multiply this by the height and divide by 24. This gives the volume of the frustum of a cone [9; pp.202-03]."

Representing this in the modern notation :

$$V = \frac{h}{24}\sqrt{10\,\{a^2 + b^2 + (a+b)^2\}^2}$$

$$V = \frac{\pi h}{24}\sqrt{\{a^2 + b^2 + (a+b)^2\}^2}$$ (taking $\sqrt{10} = \pi$)

Where for the frustum, 'h' is the height and 'a, b' are the diameters of the base and the top.

(b) Śrīdhara's formula for the approximate volume of a sphere is as follows (*Triśatikā*.56):

गोलव्यासघनार्धं स्वाष्टादशभागसंयुतं गणितम्॥

"Half the cube of the diameter of the sphere added to one-eighteenth of itself gives the volume of the sphere".

Śrīdhara's formula for the Volume of a sphere $=(d^3/2) + d^3/(2 \times 18) = (19/36)\, d^3$

6. Mahāvīrācārya

Mahāvīra was a *Digambara* Jain, who lived during the reign of a Rāṣṭrakūṭa king called, Amoghavarṣa. Mahāvīrācārya wrote his *Gaṇita-Sāra-Saṅgraha* (GSS) in 850 AD. This is regarded as a proper book containing rules and many examples. This represents the mathematics prevailed at that time and it is studied for a long time, especially by the Jains. This work treats *Khāta-vyavahāra* as a separate topic.

(a) Mahāvīra gives a rule for arriving at the cubical contents of excavations (*GSS*.VII.4):

क्षेत्रफलं वेधगुणं समखाते व्यावहारिकं गणितम्।
मुखतलयुतिदलमथ सत्संख्याप्तं स्यात्समीकरणम्॥

"The approximate measure of the cubical contents in a regular excavation is equal to the product of the area of base and depth. When the excavation is not uniform, it is separated into several units. The sums of the average of top and bottom dimensions are halved; then their sum is divided by the number of the said halved quantities. Such is the process of arriving at the average equivalent value [10; p.258]".

(b) *Gaṇita-Sāra-Saṅgraha* gives a rule for finding the volume of frustum-like solid in 3½ verses (*GSS*.VIII.9-11½):

बाह्याभ्यन्तरसंस्थिततत्क्षेत्रस्थबाहुकोटिभुवः।
स्वप्रतिबाहुसमेता भक्तास्तत्क्षेत्रगणनयान्योन्यम्॥
गुणिताश्च वेधगुणिताः कर्मान्तिकसंज्ञगणितं स्यात्।
तद्बाह्यान्तरसंस्थिततत्क्षेत्रे फलं समानीय॥
संयोज्य सङ्ख्याप्तं क्षेत्राणां वेधगुणितं च।
औण्ड्रफलं तत्फलयोर्विशेषकस्य त्रिभागेन॥
ससंयुक्तं कर्मान्तिकफलमेव हि भवति सूक्ष्मफलम्॥

The interpretation of the rules as following:

According to Gupta [11], this rule is a sort of generalization of a similar rule given by Brahmagupta.

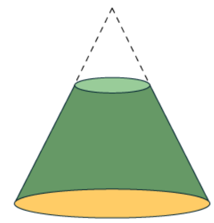


Fig. 5: Frustum

The rule is expressed here in mathematical symbols. Let the number of sections considered be 'n' and $f\,(a,b,c,...)$ denote the area of a section as a function of its defining (independent) linear dimensions $a,b,c,\ldots$. Let $a_1, b_1, c_1,\ldots;\ a_2, b_2, c_2,\ldots;\ \ldots\ a_n, b_n, c_n,\ldots$ be these linear dimensions (or sides) of the 'n' sections. Then the *Karmāntika* (Practical) volume (K) will be

$K = h \cdot f(\bar{a}, \bar{b}, \bar{c}, \ldots)$, where h is the depth of the excavation and

$\bar{a} = (a_1 + a_2 + \ldots + a_n) \div n$. etc.

The *auṇḍra* (gross) volume = $A = h\,(A_1 + A_2 + \ldots + A_n) \div n$. where $A_1 = f\,(a_1, b_1, c_1\ldots)$ etc. are the areas of 'n' sections. Then, according to the above rule, the accurate volume is given as

$$V = K + \frac{A-K}{3} = (2K + A)/3$$

Here the exact volume of the solid is obtained when the sections are circles, squares etc. the corresponding sides in various sections being parallel to each other in all the polyhedrons. If we consider only the two extreme sections (the top and the base), then 'n' will be two and the rule is same as the one given by Brahmagupta.

(c) Mahāvīra gives a rule (*GSS*.VII.19½ - 20½) for arriving at the value of the cubical content of an excavation having in the middle (of it) a tapering projection.

परिखामुखेन सहितो
विष्कम्भस्त्रिभुजवृत्तयोस्त्रिगुणात्।
आयामश्चतुरश्रे चतुर्गुणो व्याससङ्गणित:॥
सूचीमुखवद्वेधे परिखा मध्ये तु परिखार्धम्।
मुखसंहितमथो करणं प्राग्वत्तलसूचिवेधे च॥

"The breadth (of the central mass) increased by the top-breadth of the surrounding ditch, then multiplied by three, gives rise to the value of the (required) perimeter in the case of triangular and circular excavations. In the case of a quadrilateral excavation (this same value of the perimeter results) by multiplying four and breadth [10; pp.261-62]".

If '*d*' is the breadth of the central figure, '*b*' is the breadth and '*h'* is the depth of the ditch, then, length of the equilateral triangular or circular ditch = $(d + b) \times 3$

Karmāntika phala $= K = (d + b/2)(3) \times (b/2)$
Auṇḍra phala $= A = (d + b)(3) \times (b/2)$
Sūkṣma phala $= S = K + \frac{A-K}{3}$
Cubical content $= V = \{ K + \frac{A-K}{3} \} \times$ depth

(d) Mahāvīra's rule for arriving at the volume of a sphere is as follows (*GSS*.VII.28½):

व्यासार्धघनार्धगुणा नव गोलव्यावहारिकं गणितम्।
तद्दशमांशं नवगुणमशेषसूक्ष्मं फलं भवति॥

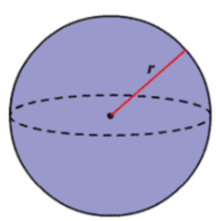


Fig. 6: Sphere

"The half of the cube of half the diameter, multiplied by nine, gives the approximate value of the cubical contents of a sphere. This (approximate value), multiplied by nine and divided by ten, on neglecting the remainder, gives rise to accurate value of the cubical measure [10; p.265]".

The diameter of the sphere = *d*

Approximate volume of the sphere =
$(1/2)\,(d/2)^3 \times 9 = (9/2)\,r^3$

Accurate volume of the sphere =
$(9/2)\,r^3 \times (9/10)$

Note: This formula is not accurate as claimed by *GSS*.

(e) Mahāvīra has given a rule for arriving at the volume of an excavation in the form of a triangular pyramid (*GSS*.VII.30½):

भुजकृतिदलघनगुणदशपदनवहृद्व्यावहारिकं गणितम्।
त्रिगुणं दशपदभक्तं शृङ्गाठकसूक्ष्मघनगणितम्॥

"The cube of half the square of the side (of the basal equilateral triangle) is multiplied by ten; and the square root (of the resulting product is) divided by nine. This gives rise to the approximately calculated value. This value when multiplied by three and divided by the square root of ten, gives rise to the accurately calculated cubical contents of the pyramidal excavation [10; pp.265-66]".

For the pyramidal excavation having base as equilateral triangle:

Approximate volume $= \sqrt{\left(\frac{a^2}{2}\right)^3 \times 10\left(\frac{1}{9}\right)}$

Accurate volume $= \frac{a^3}{18} \times \sqrt{5} \times \frac{3}{\sqrt{10}}$
$= a^3 \times \frac{\sqrt{2}}{12}$

Fig. 7: Pyramid

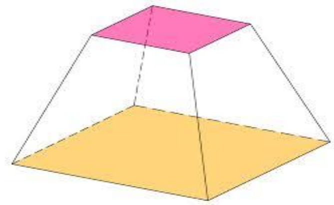


Fig. 8: Frustum

7. Bhāskarācārya II

The popular name in the history of ancient and medieval Indian astronomy and mathematics is that of Bhāskarācārya (1114 – 1193 CE). His *Siddhānta-Śiromaṇi* is studied by scholars of traditional Indian mathematics. He has perfected many of the formulae given by his ancestors. Bhāskara II treats *Khātavyavahāra* in his *Līlāvatī* as a separate topic.

(a) Bhāskara II gives the volume of an irregular ditch in *L*.214:

गणयित्वा विस्तारं बहुषु स्थानेषु तद्युतिर्भाज्या।
स्थानकमित्या सममितिरेवं दैर्घ्ये च वेधे च॥
क्षेत्रफलं वेधगुणं खाते घनहस्तसंख्या स्यात्॥

"In an irregular ditch, measure the breadth at various points, add them and divide by the number of points. This is the average breadth. Similarly, calculate the average length and depth. The product of the three averages will give the volume [12; p.151]".

(b) To find the volume of a pyramid and its frustum, a rule is given (*L*.217):

मुखजतलजतद्युतिजक्षेत्रफलैक्यं हृतं षड्भिः।
क्षेत्रफलं सममेवं वेधहतं घनफलं स्पष्टम्॥
समखातफलत्र्यंश: सूचीखाते फलं भवति॥

Sum of the areas of top and bottom faces and that of the rectangle which has the sum of their sides as sides, divided by six gives the average area; this multiplied by depth is the (average) volume. One-third of the volume of a regular equal solid becomes the volume of pyramid.

(c) An *upapatti* is given in *Buddhivilāsinī*, a commentary on *Līlāvatī* by Gaṇeśa Daivajña [*BV*. p.223] for the above rule using the example in *L*. 218:

मुखे दशद्वादशहस्ततुल्यं
विस्तारदैर्घ्यं तु तले तदर्धम्।
यस्या: सखे सप्तकरश्च वेध:
का खातसंख्या वद तत्र वाप्याम्॥

"Tell the quantity of the excavation in a tank, of which the length and breadth are equal to twelve and ten cubits at its mouth, and half as much at the bottom, and of which the depth, friend, is seven cubits [13; p.151]".

Gaṇeśa Daivajña in his *Buddhivilāsinī* (*BV*) [14], explains by providing an interpretation similar to what is done today in the engineering drawing:

अत्रोपपत्ति:। तत्र मुखे दश द्वादशेति वक्ष्यमाणखातस्य निदर्शनम्। अत्र तलक्षेत्रानुसारिण: सप्तवेधस्य समखातस्य घनफलं २१०। अवशिष्टखातस्य कोणेषु चत्वारि सूचीखातखण्डानि एवमष्टौ। तथा पूर्वादिदिक्षु

चत्वारि। दर्शनम्। कोणस्थखण्डचतुष्टययोगेन जातं सप्तवेधं सूचीखातदर्शनम्- तस्यास्य सूचीखातस्य घनफलं ७०। तथा दिक्षु स्थितखण्डचतुष्टयमध्ये द्वयोर्द्वयोरन्योन्याभिः मुखयोर्योगेन जातं सप्तवेधं समखातद्वयम्। तयोर्दर्शनम् । अनयोर्घनपले १०५।१०५। एवं चतुर्णामेषां २१०। ७०। १०५। १०५ योगेन जातं सर्वखातस्य फलं ४९०। मुखजतलजतद्युतिजेत्यादिक्रियया समं दृश्यत इति। यद्वा सामान्येनान्यथोच्यते। मुखजतलजविस्तारयोर्योगार्धं मध्यवर्ति-विस्तार:। एवं दैर्घ्ये च। तयोर्घातो मध्यफलम्। अत्रार्धेनार्धं गुणितं चतुर्थांश: स्यादिति विस्तारयोगदैघर्ययोगयोर्घात-श्चतुर्गुणं मध्यफलं स्यात्। तथा मुखजं फलमेकगुणं तथा तलजमेकगुणमेषां त्रयाणां योग: षड्गुणं समफलं स्यात्। अत उक्तं मुखजतलजेत्यादि।

Here is the *Upapatti*: "An excavation having 10 and 12 as the dimensions of the face is considered. [The corresponding sides of the base are 5 and 6]. According to this base, with depth 7, the cubical content (volume) of the uniform excavation [having base and face of dimensions 5 × 6, a rectangular parellelopiped] is 210. In the remaining excavation, in the corners there are four parts in the form of needle-shaped portions of excavations (*sūcī-khāta)* and there are four in the directions of east etc, thus there are eight parts (Fig. 10). The volume of four parts (triangular prisms) in the corners with depth 7 is 70. Similarly, among the four parts standing in the four directions, both the sides facing each other being added we get two uniform solids of depth 7. Their volumes are 105 and 105. By adding the four volumes 210, 70, 105 and 105 the volume of the full excavation is obtained as 490. Thus the rule '*Mukhajatalaja*...'(*L.* 217) is applied. It is also stated the other way. Half of the sum of the breadths of face and base is the breadth of the middle one. In the same way length is found. The product of these is the average area. Here half multiplied by half will become one fourth and so 4 times the product of the sum of the breadths and the sum of the lengths is cubical content of middle parts (*madhyaphalam*). The sum of the three volumes produced by face, base and *madhyaphalam* is six times the cubical content of the excavation".

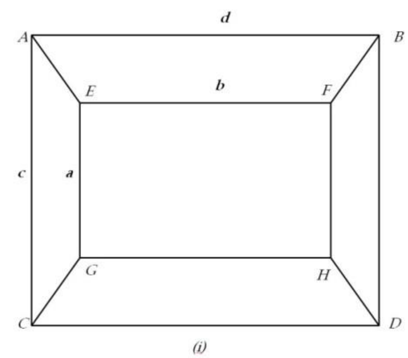


Fig. 9: Frustum

Volume (V) of a frustum with similar rectangular faces of sides, (a,b) and (c,d); depth 'h' is given in *L.* 217 by

$$V = \{ ab + cd + (a + c)(b + d)\} (h/6)$$

The *upapatti* can be explained as follows:

i. In the middle, there is a parallelopiped with sides 5 , 6 and depth 7. [Shown below in Fig. 10. (i)]. Its volume = 5×6×7 = 210.
ii. 2 *sücī-khātas* on the side 5 each with volume $7 \times \frac{5}{2} \times \frac{12-6}{2} = \frac{105}{2}$. [Shown below in Fig.10 (ii)]

iii. 2 *sūcī-khātas* on the side 6 each with volume $= 7 \times \frac{6}{2} \times \frac{10-5}{2} = \frac{105}{2}$ [shown in Fig. 10 (iii)]. Two such sections together make a parallelopiped; so each is half the volume of a parallelopiped.

iv. 4 rectangular pyramids, each with volume $= \frac{7}{3} \times \frac{12-6}{2} \times \frac{10-5}{2} = \frac{35}{2}$. [Shown in Fig.10 (iv) below]. Three such sections together make a parallelopiped; so each is one-third of the volume.

Fig. 10:

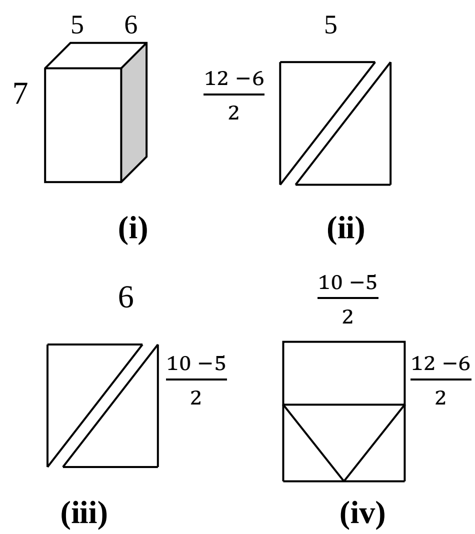


Fig. 10a (ii):

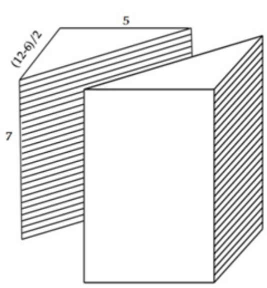
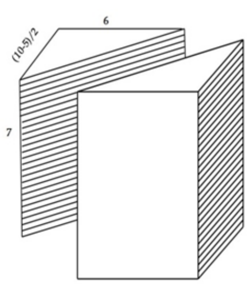


Fig. 10a (iii):



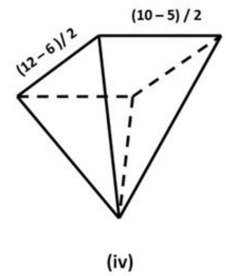


Fig. 10a (iv)

Total volume =

$210 + 2 \times \frac{105}{2} + 2 \times \frac{105}{2} + 4 \times \frac{35}{2} = 490$

BV says (a) A rectangular cuboid of sides $a = 5$ and $b = 6$ and depth $h = 7$ units inserted in a frustum with bottom base as $a \times b = 5 \times 6$ and top face $c \times d = 10 \times 12$ and depth 7 units. (b) Four triangular prisms on the 4 vertical faces of the box. (c) Four rectangular pyramids at the 4 corners of the top face of the box.

The sum of the following volumes is the volume of the frustum:

(a) Volume of rectangular box, $V_1 = abh$

(b) Volume of 4 triangular prisms, $V_2 = 2 . \frac{a}{2} . \frac{d-b}{2} . h + 2 . \frac{b}{2} . \frac{c-a}{2} . h$

(c) Volume of 4 rectangular pyramids, $V_3 = 4 . \frac{c-a}{2} . \frac{d-b}{2} . \frac{h}{3}$

Adding the 3 volumes

$V = V_1 + V_2 + V_3 = abh + 2 . \frac{a}{2} . \frac{d-b}{2} . h + 2 . \frac{b}{2} . \frac{c-a}{2} . h + 4 . \frac{c-a}{2} . \frac{d-b}{2} . \frac{h}{3}$

$= h \left(\frac{ad}{6} + \frac{bc}{6} + \frac{cd}{3} + \frac{ab}{3}\right)$

$= \frac{h}{6} (ab + cd + ad + ab + bc + cd)$

$= \frac{h}{6} \{ ab + cd + (a + c)(b + d)\}$

The volume required in the example (*L*. 218) is

$\frac{7}{6}$ { 5 × 6 + 10 × 12 + (5 + 10)(6 + 12)} $= \frac{1}{6} \times$ 2940 = 490.

The volume (*V*) of a frustum with similar rectangular faces of sides, '*a*,*b*' and '*c*,*d*'; depth '*h*' is given by

V = {*ab* + *cd* + (*a* + *c*)(*b* +*d*)} (*h*/6). Volume of a pyramid is one-third of a prism with the same base and same height.

Note: Figures 10 and 10a above, as given in *Buddhivilāsinī*, give the vertical cross-section of the excavation as looked from top and the horizontal cross section. In modern Engineering drawing these are said to be 'Plan and Elevation.'

(d) To find the volume of a ditch with square or circular base a rule is given (*L*.219a):

उच्छ्रयेण गुणितं चिते: किल क्षेत्रसंभवफलं घनं भवेत्।

'Volume of a prism is equal to the product of the area of its base and height.'

To find the volume of a heap of grain i.e. volume of a cone is given in *L*.227b:

भवति परिधिषष्ठे वर्गिते वेधनिघ्ने।
घनगणितकरा: स्युर्मागधास्ताश्च खार्य:॥

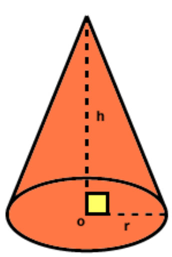


Fig. 11: Cone

"A sixth of the circumference being squared and multiplied by the depth (height). The product will be the solid cubits, and they are *khārīs* of Magadha [13; p.156]".

Volume of a cone

= height × (circumference/6)^2

= $h \times (2\pi r/6)^2 = (\pi^2 r^2 h) / 3^2$

= $(1/3)\, \pi r^2 h$ {taking π = 3}

(e) A rule to find the volume of a sphere is given in *L*.201c, d:

गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नम्।
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम्॥

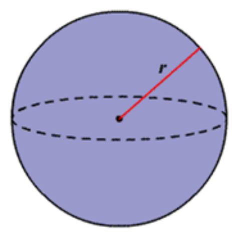


Fig. 12: Sphere

"This content of the surface of the sphere, multiplied by the diameter and divided by six is the precise solid, termed cubic, content within the sphere [13; p.136]".

The volume of a sphere = Surface area of sphere × *d*/6

Surface area is = 4 × circumference × *d*/4

⇒ The volume of sphere

= $4 \times C \times (d/4)(d/6)$

= $4 \times 2\pi r \times (1/4)\ 2\ r \times (2\ r/6)$ {in present day notation}

= $(4/3)\pi r^3$

Note: Bhāskara II has given the perfect formulae for the volume of the solids i.e. the cubical content of the excavations, compared to the earlier mathematicians.

Gaṇeśa gives *upapatti* [15; p.201] for the volume of a sphere for the rule given in *L*.201 c, d:

अथ घनफलोपपत्तिः- गोलस्य व्यासार्धतुल्यदैर्घ्याणि सूच्याकाराणि सूच्यग्राणि चतुष्कोणदैर्घ्याणि मूर्ध्नि हस्ततुल्यविस्तृतिदैर्घ्याणि कृतानि खण्डानि पृष्ठफलसंख्यान्येव भवन्ति। एवंविधैकखण्डस्य मूर्ध्नि क्षेत्रफलं रूपमेव। खण्डदैघर्यं व्यासार्धं स एव वेधः। तेन गुणितं क्षेत्रफलं तस्य त्रयंशो घनफलं स्यात्। क्षेत्रफलं वेधगुणं खाते घनहस्तसंख्या स्यात्। समखातफलत्र्यंशः सूचीखाते फलं भवतीति वक्ष्यमाणत्वात्। अतो व्यासषडंश एवैकखण्डस्य घनफलं स्यात्। तत्पृष्ठफलगुणितं सर्वगोलस्य घनफलं जायत इति। अत उक्तं - तदपि च फलं पृष्ठजमित्यादि॥

"Then the [14] *upapatti* for cubical content - Needle–shaped pointed sections (pyramids), which have length equal to half the diameter of the sphere and which have on top, squares that have length and width equal to one *hasta* (unit length), are made and the number of such sections are equal to the surface area. The area on top of such a section is one (square unit) itself. The length of a section is half the diameter, that itself is depth. One-third of the area multiplied with that will be the cubical content, as it will be said that 'the area multiplied by the depth will be the number of cubical *hastas* of an excavation' (*L*.214 ef) and that 'one-third of the (cubical) content of a regular excavation is the (cubical) content of a needle- shaped excavation'(*L*.217 ef). Therefore one sixth of diameter only is the cubical content of one section. That multiplied by the surface area becomes the cubical content of the full sphere. Hence it is said – 'This surface area also...'(*L*. 201 cd)".

This is explained: The whole surface area of the sphere is divided into unit squares. Thus there are *S* (equal to surface area) unit squares. Corresponding to each unit square, one needle- shaped section (*sücyagra*) is made. The base surface of each needle- shaped section is a square of unit length *i.e.* its area is one unit square. This has depth equal to half of the diameter.

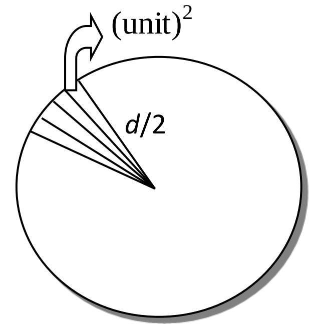


Fig. 13: Volume of sphere

The volume of 1 needle- shaped section
= base area × depth × 1/3
= 1 unit square × (d/2) x (1/3)

The volume of '*S*' needle- shaped sections = $S \times (d/6)$

This can be expressed in modern notation as = $4\pi r^2 \times (2r/6)$
= $(4\pi r^3/3)$ cubic units

Note: Here the sphere is divided into small units and then their sum is taken to find the volume. Here we see the rudiments of Calculus.

Yuktibhāṣā of Jyeṣṭhadeva (1500 – 1610 CE) derives the expressions for the formulae for volume of a sphere etc. with the help of the methods of Calculus, which were discovered in Europe later by Newton and Leibnitz (17th-18th CE).

The derivation of the volume of a sphere as in *Gaṇita-yukti-bhāṣā* [16; pp.264-66], is as follows:

Let r be the radius of the sphere and C, the circumference of a great circle.
Area of circle = (1/2) C× r (1)

The half-chord B_j is the radius of the jth slice into which the sphere has been divided. The corresponding circumference is $\left(\frac{C}{r}\right)B_j$ and from (1), the area of this circular slice is = $\frac{1}{2}\left(\frac{C}{r}\right)B_j^2$

If Δ is the thickness of the slices, then the volume of the jth slice is = $\frac{1}{2}\left(\frac{C}{r}\right)B_j^2\,\Delta$

Volume of a sphere = the sum of the squares of the Rsines B_j^2

$$V \approx \frac{1}{2}\left(\frac{C}{r}\right)[B_1^2 + B_2^2 + \cdots B_n^2]\Delta \quad (2)$$

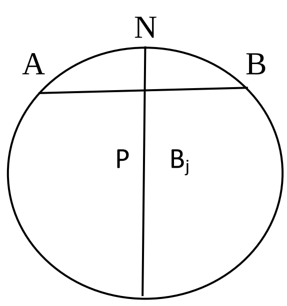


Fig. 14: Square of half-chord equal to product of *śaras*

In the Fig. 14, $AP = PB = B_j$ = jth half-chord, starting from N, the north point.

$B_j^2 = AP \times PB = NP \times SP$ (by *Āryabhaṭīya* rule, Gaṇita. 17)

$$= \frac{1}{2}\left[(NP + SP)^2 - (NP^2 + SP^2)\right]$$

$$= \frac{1}{2}\left[(2R)^2 - (NP^2 + SP^2)\right] \quad (3)$$

If $\Delta = \frac{2r}{n}$, be the thickness of each slice, the jth R versine $NP = j\,\Delta$ and its complement SP = $(n - j)\,\Delta$. Hence, while summing the squares of the Rsines B_j^2, both NP^2 and SP^2 add to the same result. Thus by (2) and (3)

$$V \approx \frac{1}{2}\left(\frac{C}{r}\right)\left(\frac{2r}{n}\right)\left(\frac{1}{2}\right)[(2r)^2 + (2r)^2 \ldots + (2r)^2]$$
$$-\left[\frac{1}{2}\left(\frac{C}{r}\right)\left(\frac{2r}{n}\right)\left(\frac{1}{2}\right)\left(\frac{2r}{n}\right)(2)[(1)^2 + (2)^2 \ldots + (n)^2]\right.$$
(4)

For large n, the sum of the squares (*varga-saṅkalita*) is essentially one-third the cube of the number of terms. Then (4) becomes

$$V = \left(\frac{C}{2r}\right)\left(4r^3 - \frac{8r^3}{3}\right)$$
$$= \left(\frac{C}{6}\right)d^2$$

Hence the Volume of sphere = one-sixth of circumference × square of diameter, which is same as V = $\frac{4}{3}\pi r^3$

This is the proof given by *Gaṇita-yukti-bhāṣā* for the rule *L*.201 cd.

Conclusion

The *Khāta-vyavahāra* or the excavations and cubical content of solids, is treated as a separate topic by the mathematicians in the ancient Indian texts starting from *Śulabasūtras*, *Āryabhaṭīya*, and so on up to *Līlāvatī* of 12th century CE. The important rules of *Khāta-vyavahāra,* which are dealt with by the ancient scholars have been discussed here. Āryabhaṭa II, Nemicandra, Śrīpati and Nārāyaṇa Paṇḍita mostly repeat the earlier mathematicians' rules. The formulae for prisms, pyramids, frustum, cone and sphere have evolved from approximate calculations up to accurate formulae, which are used even today. These formulae for finding the volume are used in the construction of temples and buildings and are used at present, by the *sthapatis*, who follow the traditional methods. Calculating the earth taken from an excavation or pit, that is going to be dug, helps to plan the use of the earth or mud dug such as whether that can be used to build an embankment, build a platform, fill up some other pit or use it for the other constructions.

In the Times of India (Kolkata, August 17, 2014, p.10), there appeared an interview with Manjul Bhargava FRS, who was awarded the Fields Medal in 2014.

- When asked: Why is India still a middle power in mathematics despite its famed legacy? Manjul in his reply said:

'Students in India should be taught about the great ancient Indian legacy of mathematicians, since ancient times, like Panini, Pingala, Hemachandra, Aryabhata, Bhaskara, Brahmagupta, Madhava for example and more recently Ramanujan etc. Their stories and works inspired me, and I think they would inspire students across India. Many of these works were written in Indian languages in beautiful poetry, with a flavour of Indian stories and contain some of the most important breakthroughs in the history of mathematics. '

The above statement of Manjul Bhargava reveals the importance and relevance of ancient Indian mathematics in the Sanskrit texts and how the study of these texts can create enthusiasm among young students and influence them for research. In this topic especially the study of three-dimensional solids and the beginning of calculus are appealing to the students.

Future work: Only a few books have been edited from manuscripts and translated so far. If more works from the vast repository of manuscripts are edited and published, then more of the hidden wisdom will come to light. This ancient knowledge of Indian mathematicians is to be imparted to young children.

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